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Unstable relativistic quantum fields: two models

I E Antoniou[1,2], **M Gadella**[1,3], **J Mateo**[3] **and G P Pronko**[1,4]

[1] International Solvay Institutes for Physics and Chemistry, CP 231, Campus Plaine ULB, Bd du Triomphe, 1050 Brussels, Belgium
[2] Theoretische Natuurkunde, Free University of Brussels, Belgium
[3] Departamento de Física Teórica, Facultad de Ciencias, E-47011, Valladolid, Spain
[4] Institute for High Energy Physics, Protvino, 142284 Moscow region, Russia

E-mail: gadella@fta.uva.es

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Abstract

We present two models of relativistic interactions in quantum mechanics that produce resonances. In both cases, these resonances are described by poles of the analytic continuations of Green functions in terms of the variable energy. We develop the first model in detail and motivate and describe the second one. We compare the common features between these two models. The analysis is made in the context of quantum field theory.

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1. Introduction

We have recently studied [1] an exactly solvable model for relativistic unstable quantum fields. The model consists of a local scalar field φ in quadratic interaction with a bilocal scalar field ψ with mass spectrum $[M, \infty)$. The resonance manifests itself as a pair of simple poles on the analytic continuations, through the mass spectrum of ψ, of the Green function $G(E, \mathbf{k})$ on the energy E. Corresponding to these two resonance poles, we have constructed their Gamow vectors, as eigenvectors of the energy operator P_0 with eigenvalues given by the resonant poles [1]. Since the poles are complex, the solution of the spectral equation involving them cannot be found on the usual Fock space, but in two locally convex extensions of the Fock space, one for each extension of $G(E, \mathbf{k})$. This can be done using the theory of rigged Hilbert spaces [2].

We have shown moreover that the unitary group given by the evolution operator $\exp(-\mathrm{i}tP_0)$ as in the case of nonrelativistic quantum mechanics [3–5] splits into two distinct semigroups. This splitting confirms that the proposed model is intrinsically irreversible in the sense of Prigogine [6].

The question about the physical content of an unstable field theory is similar to the same question in other formalisms such as, say, QCD. Indeed in these theories, we can construct the

Green functions with the help of the usual Feynman diagrams. However, we cannot construct the elements of the S-matrix reasonably as we cannot obtain the space of true asymptotic states. If we use as this space the naive Fock space of the fundamental fields which enter into the Lagrangian, we obtain the infrared divergences in, e.g., QED or QCD. This is a signal that the appropriate Green functions do not have the corresponding singularities on the mass shell of the fundamental fields [7]. Here, we discuss the similarities of these questions for the unstable field theory and the theories with massless fields and give the solution of the problem of the definition of the space of true asymptotic states for unstable field theory.

Thus, the objective of the present paper is to provide more insight into the theory of unstable relativistic quantum fields. Here, we shall present a new model of unstable interaction between relativistic quantum fields which can be exactly solvable in one sector. We also shall recall the example developed in [1] in order to compare it with the other model and complete the discussion and motivation of this example, which is not included in [1].

The main object of conventional field theory is the Green function of interacting fields, which is given by

$$G(x_1, \dots, x_n) = \langle 0|T\left\{\varphi_{i_1}(x_1)\varphi_{i_2}(x_2)\cdots\varphi_{i_n}(x_n)\right\}|0\rangle \tag{1}$$

where $|0\rangle$ is the vacuum, T the time ordering operator and the $\varphi_i(x)$ are Heisenberg fields. In the interaction picture (Gell-Mann–Low representation), one has

$$G(x_1, \dots, x_n) = \langle 0|T\left\{\varphi_{i_1}(x_1)\varphi_{i_2}(x_2)\cdots\varphi_{i_n}(x_n)S\right\}|0\rangle \tag{2}$$

where S is the operator defined by

$$S := T\exp\left\{\int_{-\infty}^{\infty} \mathrm{d}t\, H_{\mathrm{int}}(t)\right\}. \tag{3}$$

The Fourier transform of the Green function $G(x_1, \dots, x_n)$ is

$$\delta\left(\sum_{j=1}^{n} p_j\right) G(p_1\cdots p_n) = \int_{\mathbb{R}^{4n}} \mathrm{d}x_1\cdots\mathrm{d}x_n \exp\left(\mathrm{i}\sum_{j=1}^{n} p_j\cdot x_j\right) G(x_1, \dots, x_n). \tag{4}$$

The δ-function on the left-hand side of (4) reflects the energy–momentum conservation and it arises because of the translation invariance of $G(x_1, \dots, x_n)$, i.e.,

$$G(x_1, \dots, x_n) = G(x_1 + a, \dots, x_n + a) \tag{5}$$

where a is an arbitrary 4-vector.

The elements of the S-matrix can be expressed via Green functions with the help of the so-called Lehman–Symanzik–Zimmermann reduction formalism

$$\begin{aligned} S(p_1\cdots p_n) &= \left(p_1^2 - m_1^2\right)\left(p_2^2 - m_2^2\right)\cdots\left(p_n^2 - m_n^2\right)G(p_1\cdots p_n) \\ &\left\{p_i^2 \longrightarrow m_i^2\right\}. \end{aligned} \tag{6}$$

In other words, the S-matrix elements are the residues of the Green functions on the mass shell $\left(p_i^2 = m_i^2\right)$.

As a result, the S-matrix in the Fock space exists if and only if the Green functions $G(p_1\cdots p_n)$ have only pole singularities with respect to all variables p_i^2 at the points m_i^2 (which do not need to be identical to the bare masses of the particles).

If the Green functions have another kind of singularity (double pole, cut, etc) we *cannot define the S-matrix in the Fock space*. In other words, if the Green functions do not have a pole singularity with respect to the variables p_i^2, by calculating the residue formally we would obtain the zero result, i.e., the S-matrix would be identically zero. In perturbation expansions, however, this zero may be seen as a divergence. In fact, infrared divergences are manifestations of this kind of phenomenon.

An analysis of the Fourier transform of $G(p_1 \cdots p_n)$ with pole singularities with respect to the variables p_i^2 leads to the conclusion that for large spacetime separation ($x_i - x_j \longmapsto \infty$), the Green function becomes

$$G(x_1, \ldots, x_n) \longmapsto \exp\left(\sum_{j=1}^{n} \mathrm{i}k_j \cdot x_j\right) \tag{7}$$

where $k_i^2 = m_i^2$. Therefore, for large spacetime separation, the Green function becomes the solution of the free field equation with respect to each coordinate, i.e., we have free evolution. This implies that, in order that the Green function has the correct pole singularity (so that the S-matrix does exist), the interaction should decrease sufficiently fast.

The difficulties in the S-matrix definition in the Fock space can also be seen in the time-dependent Rayleigh–Schrödinger perturbation theory:

$$S := U(\infty, -\infty) \tag{8}$$

where

$$U(t_1, t_2) = T \exp\left[-\mathrm{i} \int_{t_2}^{t_1} \mathrm{d}t\, V(t)\right]. \tag{9}$$

Here, $V(t)$ is the perturbation in the interaction representation, with respect to the usual separation of the total Hamiltonian

$$H = H_0 + V. \tag{10}$$

The S-matrix defines the transition amplitude T by

$$S_{\alpha\beta} = \delta_{\alpha\beta} + 2\pi \mathrm{i}\delta(E_\beta - E_\alpha)T_{\alpha\beta}. \tag{11}$$

From the definition of the S-matrix, we have

$$S = 1 + (-\mathrm{i}) \int_{-\infty}^{\infty} \mathrm{d}t\, V(t) + (-\mathrm{i})^2 \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\, V(t_1)V(t_2) + \cdots. \tag{12}$$

The time dependence of $V(t)$ in the interaction picture is given by

$$\langle a|V(t)|b\rangle = \exp(\mathrm{i}(E_a - E_b)t)\langle a|V(0)|b\rangle \tag{13}$$

where $|a\rangle$ is the eigenvector of H_0 with eigenvalue E_a. Thus, the transition amplitudes are

$$T_{ba} = V_{ba} + \int \mathrm{d}c \frac{V_{bc}V_{ca}}{E_a - E_c + \mathrm{i}\epsilon} + \int \mathrm{d}c_1 \int \mathrm{d}c_2 \frac{V_{bc_2}V_{c_2c_1}V_{c_1a}}{\left(E_a - E_{c_1} + \mathrm{i}\epsilon\right)\left(E_a - E_{c_2} + \mathrm{i}\epsilon\right)} + \cdots. \tag{14}$$

Here

$$V_{ba} := \langle b|V(0)|a\rangle \tag{15}$$

and the integration over intermediate states also involves summation over discrete quantum numbers.

The denominators in equation (14) arise due to the integration over an infinite interval of time. Sometimes these denominators may become small, giving rise to divergences of some matrix elements of T. These divergences are obviously connected with a slow decrease of $V(t)$. From the point of view of the expansion of T_{ba}, we may also say that these small denominators are the manifestation of degeneracies of intermediate states with initial or final states. There are two types of such dangerous degeneracies:

(i) the degeneracy due to zero photon mass gives rise to the so-called infrared divergences in QED [8].

(ii) the degeneracy due to broken stability conditions. Here we mean the following: consider an unstable particle with mass M. Its energy will be given by $(M^2+\mathbf{k}^2)^{1/2}$. Suppose that field theory allows this particle to decay into two other particles with masses m_1 and m_2. For that we need the energy momentum balance:

$$(M^2+\mathbf{k}^2)^{1/2} = \left(m_1^2+\mathbf{k}_1^2\right)^{1/2} + \left(m_2^2+\mathbf{k}_2^2\right)^{1/2} \qquad \mathbf{k}=\mathbf{k_1}+\mathbf{k}_2.$$

This problem leads to the famous so-called *Poincaré problem of small denominators* [9, 10] and in this paper we are going to give a detailed example: how we have to modify the scattering theory in order to overcome this difficulty.

2. The model and its asymptotic dynamics

Most models of interaction in quantum field theory cannot be solved exactly but only approximately. Thus the question arises whether the above-mentioned formalism can be applied to situations in which approximate solutions to field equations are all that we can have. In order to see that the answer must be positive, we propose the following model involving two real scalar relativistic quantum fields $\varphi(x)$ and $\psi(x)$, with respective masses m and M, coupled with the simplest cubic interaction. Thus, the Hamiltonian of the system is given by

$$H = H_m + H_M + V \tag{16}$$

where

$$H_M = \int \mathrm{d}x(\dot{\psi}^2 + (\nabla\psi)^2 + M^2\psi^2) \tag{17}$$

$$H_m = \int \mathrm{d}x(\dot{\varphi}^2 + (\nabla\varphi)^2 + m^2\varphi^2) \tag{18}$$

$$V = \lambda \int \mathrm{d}x\, \psi(x)\varphi^2(x). \tag{19}$$

The dot means first derivative with respect to time. By boldface letters we denote three-dimensional vectors. Four-dimensional vectors in Minkowski space are denoted by roman style letters. The products of two 4-vectors as well as the scalar products of two 3-vectors are denoted by a dot. We use the standard metric (+, −, −, −) of Minkowski space. As an illustration of our notation, we write: $k\cdot x = k_0x_0 - \mathbf{k}\cdot\mathbf{x}$.

In terms of the creation and annihilation operators, the above formulae can be written in the interaction picture as

$$H_m = \int \mathrm{d}\tilde{\mathbf{q}}\, \rho(\mathbf{q}) b^\dagger(\mathbf{q}) b(\mathbf{q}) \tag{20}$$

$$H_M = \int \mathrm{d}\tilde{\mathbf{k}}\, \omega(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k}) \tag{21}$$

$$\begin{aligned} V_I(t) = \lambda \int \mathrm{d}\tilde{\mathbf{k}}\,\mathrm{d}\tilde{\mathbf{q}}_1\,\mathrm{d}\tilde{\mathbf{q}}_2 \int \mathrm{d}^3\mathbf{x}[a^\dagger(\mathbf{k})\exp(\mathrm{i}k\cdot x) + a(\mathbf{k})\exp(-\mathrm{i}k\cdot x)][b^\dagger(\mathbf{q}_1)\exp(\mathrm{i}q_1\cdot x) \\ + b(\mathbf{q}_1)\exp(-\mathrm{i}q_1\cdot x)][b^\dagger(\mathbf{q}_2)\exp(\mathrm{i}q_2\cdot x) + b(\mathbf{q}_2)\exp(-\mathrm{i}q_2\cdot x)]. \end{aligned} \tag{22}$$

The quantum field operators in (17)–(19) are

$$\begin{aligned} \psi(\mathbf{x},t) &= \int \mathrm{d}\tilde{\mathbf{k}}[a^\dagger(\mathbf{k})\exp(\mathrm{i}k\cdot x) + a(\mathbf{k})\exp(-\mathrm{i}k\cdot x)] \\ \varphi(\mathbf{x},t) &= \int \mathrm{d}\tilde{\mathbf{q}}[b^\dagger(\mathbf{q})\exp(\mathrm{i}q\cdot x) + b(\mathbf{q})\exp(-\mathrm{i}q\cdot x)] \end{aligned} \tag{23}$$

with the Lorentz invariant measure

$$\begin{aligned} \mathrm{d}\tilde{\mathbf{k}} &= \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})} \qquad \omega(\mathbf{k}) = (\mathbf{k}^2 + M^2)^{1/2} \\ \mathrm{d}\tilde{\mathbf{q}} &= \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2\rho(\mathbf{q})} \qquad \rho(\mathbf{q}) = (\mathbf{q}^2 + m^2)^{1/2}. \end{aligned} \tag{24}$$

The creation and annihilation operators in (20)–(22) satisfy the usual commutation relations

$$\begin{aligned} [a(\mathbf{k}), a^\dagger(\mathbf{k}')] &= (2\pi)^3 2\omega(\mathbf{k})\delta(\mathbf{k} - \mathbf{k}') \\ [b(\mathbf{q}), b^\dagger(\mathbf{q}')] &= (2\pi)^3 2\rho(\mathbf{q})\delta(\mathbf{q} - \mathbf{q}'). \end{aligned} \tag{25}$$

Scattering theory applies if asymptotic completeness holds true between the free and the interacting fields [11]. If this assumption is not valid one tries to find another solvable evolution which satisfies the asymptotic condition and re-establish scattering theory as a comparison between the interaction field and the redefined asymptotic field. A typical example is the Faddeev–Kulish [12] removal of infrared divergences.

After integration over the three-dimensional space on the rhs of (22) we obtain eight-terms of products of creation and annihilation operators with t-dependent exponents accomplished with three-dimensional δ-functions of momentum conservation. According to the Riemann–Lebesgue lemma [13], the asymptotic behaviour of $V_I(t)$ is defined by the behaviour of these t-dependent exponents in the integration domain. For example, one of the terms on the rhs of (22) is

$$\lambda \int \mathrm{d}\tilde{\mathbf{k}}\, \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2\, a^\dagger(\mathbf{k}) b^\dagger(\mathbf{q}_1) b^\dagger(\mathbf{q}_2) \exp(\mathrm{i}[\omega(\mathbf{k}) + \rho(\mathbf{q}_1) + \rho(\mathbf{q}_2)]t)(2\pi)^3 \delta(\mathbf{k} + \mathbf{q}_1 + \mathbf{q}_2). \tag{26}$$

The asymptotic behaviour of the integral (26) as $t \to \pm\infty$ is determined by the term $[\omega(\mathbf{k}) + \rho(\mathbf{q}_1) + \rho(\mathbf{q}_2)]$. Note that

$$\omega(\mathbf{k}) + \rho(\mathbf{q}_1) + \rho(\mathbf{q}_2)|_{\mathbf{k}+\mathbf{q}_1+\mathbf{q}_2=\mathbf{0}} \geqslant M + 2m > 0. \tag{27}$$

The integral (26) goes rapidly to zero due to the fast oscillations (indeed to the Riemann–Lebesgue lemma). Let us consider another term

$$\int \mathrm{d}\tilde{\mathbf{k}}\, \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2\, a^\dagger(\mathbf{k}) b(\mathbf{q}_1) b(\mathbf{q}_2) \exp(\mathrm{i}[\omega(\mathbf{k}) - \rho(\mathbf{q}_1) - \rho(\mathbf{q}_2)]t)(2\pi)^3 \delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2). \tag{28}$$

Again, the asymptotic behaviour of (28) is defined by the quantity

$$\Delta(\mathbf{k}, \mathbf{q}_1) = \omega(\mathbf{k}) - \rho(\mathbf{q}_1) - \rho(\mathbf{q}_2)|_{\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2=0}. \tag{29}$$

The values of $\Delta(\mathbf{k}, \mathbf{q}_1)$ depend on the masses of the fields φ and ψ. If $M < 2m$, then $\Delta(\mathbf{k}, \mathbf{q}_1)$ is strictly negative:

$$\Delta(\mathbf{k}, \mathbf{q}_1) \leqslant 2\left(m^2 + \frac{\mathbf{k}^2}{4}\right)^{1/2} - (M^2 + \mathbf{k}^2)^{1/2} < 0. \tag{30}$$

To show (30), we obtain the maximum of $\Delta(\mathbf{k}, \mathbf{q}_1)$ on $\mathbf{q}_1$ for each $\mathbf{k}$. This maximum is at $\mathbf{q}_1 = \mathbf{k}/2$ (it is certainly a maximum) and the value of $\Delta(\mathbf{k}, \mathbf{q}_1)$ at the maximum is

$$2\left(m^2 + \frac{\mathbf{k}^2}{4}\right)^{1/2} - (M^2 + \mathbf{k}^2)^{1/2}. \tag{31}$$

Note that if $M < 2m$, (31) is indeed smaller than zero. Thus, $\Delta(\mathbf{k}, \mathbf{q}_1)$ has constant sign and therefore the Riemann–Lebesgue lemma applies. As a consequence, the integral (28) decreases fast as $t \longmapsto \pm\infty$.

In contrast, let us assume that $M > 2m$. Then the maximum (31) is bigger than zero. However, for fixed $\mathbf{k}$ the term

$$\Delta(\mathbf{k}, \mathbf{q}_1) = (M^2 + \mathbf{k}^2)^{1/2} - \left(m^2 + \mathbf{q}_1^2\right) - (m^2 + (\mathbf{q}_1 - \mathbf{k})^2)^{1/2} \tag{32}$$

is obviously smaller than zero for high values of $|\mathbf{q}_1|$. Let us consider the manifold for which

$$\Delta(\mathbf{k}, \mathbf{q}_1) = 0. \tag{33}$$

The integral (28) over this region does not vanish asymptotically and the same behaviour can be observed for its complex conjugate. In other words, by relinquishing the stability condition $M < 2m$, we obtain an unstable field theory where the asymptotic condition for scattering theory fails. Therefore, following the standard procedure, we redefine the asymptotic evolution in the interaction picture as follows:

$$h_{\text{as}} = H_0 + V_{\text{as}}(t) \tag{34}$$

where V_{as} includes the slowly decreasing part of V_I. There is an ambiguity in the definition of V_{as} and we shall take it as the sum of (28) and its complex conjugate, so that V_{as} is Hermitian. This is the simplest choice for V_{as}:

$$\begin{aligned} V_{\text{as}}(t) = \lambda \int & \mathrm{d}\tilde{\mathbf{k}}\, \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2 [a^\dagger(\mathbf{k}) b(\mathbf{q}_1) b(\mathbf{q}_2) \exp(\mathrm{i}\Delta(\mathbf{k}, \mathbf{q}_1) t) \\ & + a(\mathbf{k}) b^\dagger(\mathbf{q}_1) b^\dagger(\mathbf{q}_2) \exp(-\mathrm{i}\Delta(\mathbf{k}, \mathbf{q}_1) t)] (2\pi)^3 \delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) \\ = \lambda \int & \mathrm{d}^3\mathbf{x} [\psi^+(x) \varphi^{(-)2}(x) + \psi^{(-)}(x) \varphi^{(+)2}(x)] \end{aligned} \tag{35}$$

where Δ was defined in (29) and $\varphi^{(+)}, \varphi^{(-)}$ denote the positive and negative frequency parts of φ. Same for ψ.

We would like to make a few comments here.

(1) The above comments have an obvious physical interpretation: the decay process given by

$$\psi \longmapsto \varphi + \varphi$$

is only possible if the mass M of the ψ-particle is bigger than the mass $2m$ corresponding to two φ-particles.

(2) *Renormalization*. As we shall see below the h_{as} produces a ultraviolet divergence in the equation for its eigenstates. In order to remove this divergence we add to h_{as} the appropriate counterterm

$$H_{\text{c.t.}} = \frac{1}{2} \int \mathrm{d}^3 x\, \delta M^2 \varphi^{(+)}(x) \varphi^{(-)}(x) = \int \mathrm{d}\mathbf{k} \frac{\delta M^2}{\omega(\mathbf{k})} a^\dagger(\mathbf{k}) a(\mathbf{k}) \tag{36}$$

where the mass renormalization δM^2 is of order λ^2. The appearance of ultraviolet counterterms is due to our choice of asymptotic interaction. We can of course introduce some smooth cut off in V_{as}, but it will involve additional parameters to the asymptotic states and it will generally break the relativistic invariance of the system.

(3) *Relativistic invariance*. The asymptotic system should be constructed so that it defines a representation of the Poincaré group given by the generators $\left(P_{\text{as}}^\mu, J_{\text{as}}^{\mu\nu}\right)$. Here H_{as}, P_{as}^k and $J_{\text{as}}^{\mu\nu}$ are the generators of the Poincaré algebra [11]. Needless to say, in general, an arbitrary choice for H_{as} does not respect the Lorentz invariance.

The construction of the asymptotic form of the Poincaré generators could be done perturbatively with respect to the coupling parameter λ.

(4) *Connection with realistic models*. The quantum field theory we are considering here is a simplified version of the more realistic standard model. In the standard model, the Z-boson and the W-boson become unstable, due to many open decay channels. In our model however we restrict ourselves to scalar particles.

In order to obtain the desired formula for the asymptotic Hamiltonian, we add h_{as} and $H_{\mathrm{c.t.}}$:

$$H_{\mathrm{as}}(t) = h_{\mathrm{as}}(t) + H_{\mathrm{c.t.}}. \tag{37}$$

In the Schrödinger picture, the time-dependent exponents $\exp(\pm \mathrm{i}\Delta(k, q_1)t)$ in equation (35) disappear. Now, we shall consider the eigenstates of $H_{\mathrm{as}} = H_{\mathrm{as}}(0)$, which is a Hermitian operator in the Fock space corresponding to the creation and annihilation operators satisfying (25).

From now on, we shall work in the Schrödinger picture, in order to make calculations easier. In the Schrödinger picture the total Hamiltonian H will be the following:

$$\begin{aligned} H_{\mathrm{as}} = H_M + H_m + H_{\mathrm{c.t.}} + \lambda \int \mathrm{d}\tilde{\mathbf{k}}\, \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2 [a^\dagger(\mathbf{k}) b(\mathbf{q}_1) b(\mathbf{q}_2) \\ + a(\mathbf{k}) b^\dagger(\mathbf{q}_1) b^\dagger(\mathbf{q}_2)] (2\pi)^3 \delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2). \end{aligned} \tag{38}$$

3. The solutions of the eigenvalue problem

We shall show how the above problem of an unstable quantum field can be solved exactly on a sector. Of course, as the interaction is not quadratic, no exact solution is known for all sectors. This sector corresponds to the decay of one ψ-particle into two φ-particles. The Hilbert space of this system is given by

$$\mathcal{H} := \mathcal{H}_\varphi \otimes \mathcal{H}_\varphi \oplus \mathcal{H}_\psi. \tag{39}$$

The eigenspaces of the operator

$$N = N_\psi + 2N_\varphi \tag{40}$$

are invariant subspaces of H_{as}. The operators N_ψ and N_φ are the number operators of ψ- and φ-particles. The asymptotic interaction does not affect the vacuum state and the one ψ-particle state, which are the only nondegenerate eigenspaces of N. The first degenerate case is the eigenspace of N with eigenvalue 2, which is precisely $\mathcal{H}$ as given in (39). In the subspace (39) the term V_{as} in H_{as} produces transitions of two φ-particle states to one ψ-particle and vice versa. We should mention that not every two φ-particle states will mix with one ψ-particle state, but only with those in which the two φ-particles are in the S-wave state. The three-dimensional rotation and translation invariance, which is respected by H_{as}, permits us to write the two φ-particle S-wave state in the form

$$|\epsilon, \mathbf{k}\rangle = \int \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2\, \delta(\epsilon - \rho(\mathbf{q}_1) - \rho(\mathbf{q}_2)) \delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) |\mathbf{q}_1 \mathbf{q}_2\rangle \tag{41}$$

where

$$|\mathbf{q}_1, \mathbf{q}_2\rangle = \tfrac{1}{\sqrt{2}} b^\dagger(\mathbf{q}_1) b^\dagger(\mathbf{q}_2) |0\rangle \tag{42}$$

and $\tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2$ are defined in (24). The 'δ-function normalization' for $|\epsilon, \mathbf{k}\rangle$ is

$$\langle \mathbf{k}', \epsilon' | \mathbf{k}, \epsilon\rangle = \delta^3(\mathbf{k} - \mathbf{k}') \delta(\epsilon - \epsilon') \cdot \tau(\epsilon^2 - \mathbf{k}^2). \tag{43}$$

The function τ is defined on the halfline $\epsilon > (4m^2 + \mathbf{k}^2)^{1/2}$ by

$$\begin{aligned}\tau(\epsilon^2 - \mathbf{k}^2) &= \int \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2\, \delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2)\delta(\epsilon - \rho(\mathbf{q}_1) - \rho(\mathbf{q}_2)) \\ &= \frac{1}{4(2\pi)^5}\left(1 - \frac{4m^2}{\epsilon^2 - \mathbf{k}^2}\right)^{1/2} \theta(\epsilon - (4m^2 + \mathbf{k}^2)^{1/2}) \qquad (44)\end{aligned}$$

where $\theta(x)$ is the usual Heaviside step function.

The most general linear combination of two φ-particles in the S-wave state with one ψ-particle can now be written as

$$\Phi(E, \mathbf{k}) = f(E, \mathbf{k})|\mathbf{k}\rangle + \int \mathrm{d}E'\, f(E, E', \mathbf{k})|E', \mathbf{k}\rangle \qquad (45)$$

where $|\mathbf{k}\rangle$ is the one ψ-particle state. The equation

$$(H_{\mathrm{as}} - E)\Phi(E, \mathbf{k}) = 0 \qquad (46)$$

gives a system of two equations for the functions $f(E, \mathbf{k})$ and $f(E, E', \mathbf{k})$, where

$$H_{\mathrm{as}} = H_1 + V_{\mathrm{as}}(t) \qquad (47)$$

with

$$H_1 := H_M + H_m + H_{\mathrm{c.t.}} \qquad (48)$$

and use for $H_{\mathrm{c.t.}}$ the expression given at the right in (36).

Our objective is to obtain the equations for $f(E, \mathbf{k})$ and $f(E, E', \mathbf{k})$. Noting that

$$a(\mathbf{k}')|\mathbf{k}\rangle = (2\pi)^3\delta(\mathbf{k} - \mathbf{k}')\omega(\mathbf{k})|0\rangle \qquad (49)$$

we obtain that

$$(H_1 - E)|\mathbf{k}\rangle = \left(\omega(\mathbf{k}) - E + \frac{\delta M^2}{\omega(\mathbf{k})}\right)|\mathbf{k}\rangle. \qquad (50)$$

It remains to calculate the action of $V_{\mathrm{as}}(t)$ on $|\mathbf{k}\rangle$. As we are working in the Schrödinger representation, we shall use for $V_{\mathrm{as}}(t)$ the expression written in the second line of (38). Observe that this expression is a sum of two terms. The first term contains the annihilation operators $b(\mathbf{q}_1)$ and $b(\mathbf{q}_2)$ that give a vanishing contribution when applied to $|\mathbf{k}\rangle$. The second term includes the operators $a(\mathbf{k})$, $b^\dagger(\mathbf{q}_1)$ and $b^\dagger(\mathbf{q}_2)$ and therefore gives a nonvanishing contribution on $|\mathbf{k}\rangle$. Taking (49) into account, we have for $V_{\mathrm{as}}(t)|\mathbf{k}\rangle$:

$$\begin{aligned}&\sqrt{2}\lambda \int \mathrm{d}\tilde{\mathbf{k}}'\, \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2 \frac{1}{\sqrt{2}} b^\dagger(\mathbf{q}_1) b^\dagger(\mathbf{q}_2)(2\pi)^6\delta(\mathbf{k}' - \mathbf{q}_1 - \mathbf{q}_2)\delta(\mathbf{k} - \mathbf{k}')\omega(\mathbf{k})|0\rangle \\ &\quad = \sqrt{2}\lambda \int \mathrm{d}\tilde{\mathbf{k}}\, \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2 (2\pi)^6\delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2)\omega(\mathbf{k})|\mathbf{q}_1, \mathbf{q}_2\rangle \\ &\quad = \frac{\lambda\sqrt{2}}{2}(2\pi)^3 \int \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2\, \delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2)|\mathbf{q}_1, \mathbf{q}_2\rangle \qquad (51)\end{aligned}$$

where in the second identity in (51), we have made use of (24). Now, let us take (41) and integrate it over the values of the energy. Then, we have

$$\begin{aligned}\int |E', \mathbf{k}\rangle\, \mathrm{d}E' &= \int \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2\, \mathrm{d}E'\delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2)\delta(E' - \rho(\mathbf{q}_1) - \rho(\mathbf{q}_2))|\mathbf{q}_1, \mathbf{q}_2\rangle \\ &= \frac{\lambda\sqrt{2}}{2}(2\pi)^3 \int \mathrm{d}\tilde{\mathbf{q}}_1\, \mathrm{d}\tilde{\mathbf{q}}_2\, \delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2)|\mathbf{q}_1, \mathbf{q}_2\rangle. \qquad (52)\end{aligned}$$

Comparing (51) and (52), we conclude that

$$V_{\mathrm{as}}(t)|\mathbf{k}\rangle = \frac{\lambda\sqrt{2}}{2}(2\pi)^3 \int |E', \mathbf{k}\rangle \, \mathrm{d}E'. \tag{53}$$

Now, let us apply $H_{\mathrm{as}} - E$ to the integral term of (45). It is straightforward to prove that

$$H_m|E', \mathbf{k}\rangle = E'|E', \mathbf{k}\rangle \tag{54}$$

and

$$H_M|E', \mathbf{k}\rangle = \mathbf{0}. \tag{55}$$

The proof for the eigenvalue equations (54) and (55) is based on commutation relations (25) and the form (41), (42) of $|E', \mathbf{k}\rangle$.

The next step is to apply V_{as} to the integral term in (45). Obviously, the term of V_{as} containing $a(\mathbf{k})b^\dagger(\mathbf{q}_1)b^\dagger(\mathbf{q}_2)$ gives zero when applied to this term, due to the presence of the annihilation operator $a(\mathbf{k})$. The other term gives

$$\begin{aligned} \lambda \int & \mathrm{d}\tilde{\mathbf{k}}' \, \mathrm{d}\tilde{\mathbf{q}}_1 \, \mathrm{d}\tilde{\mathbf{q}}_2 \, \mathrm{d}E' (2\pi)^3 \delta(\mathbf{k}' - \mathbf{q}_1 - \mathbf{q}_2) a^\dagger(\mathbf{k}') b(\mathbf{q}_1) b(\mathbf{q}_2) \\ & \times f(E, E', \mathbf{k}) \, \mathrm{d}\tilde{\mathbf{q}}_1' \, \mathrm{d}\tilde{\mathbf{q}}_2' \delta(E' - \rho(\mathbf{q}_1') - \rho(\mathbf{q}_2')) \\ & \times \delta(\mathbf{k} - \mathbf{q}_1' - \mathbf{q}_2') \frac{1}{\sqrt{2}} b^\dagger(\mathbf{q}_1') b^\dagger(\mathbf{q}_2')|0\rangle. \end{aligned} \tag{56}$$

Commutation relations (25) give

$$\begin{aligned} b(\mathbf{q}_1)b(\mathbf{q}_2)b^\dagger(\mathbf{q}_1')b^\dagger(\mathbf{q}_2') &= (2\pi)^6 4\rho(\mathbf{q}_1)\rho(\mathbf{q}_2)[\delta(\mathbf{q}_1 - \mathbf{q}_1')\delta(\mathbf{q}_2 - \mathbf{q}_2') \\ & \quad + \delta(\mathbf{q}_1 - \mathbf{q}_2')\delta(\mathbf{q}_2 - \mathbf{q}_1')] + T \end{aligned} \tag{57}$$

where T denotes the sum of all terms including a destruction operator to the right so that, when it acts on the vacuum state $|0\rangle$, it gives the zero vector. With this in mind, along with the definitions (24) and the fact that $a^\dagger(\mathbf{k}|0\rangle = |\mathbf{k}\rangle$, equation (56) gives

$$\begin{aligned} \frac{\lambda}{\sqrt{2}} \int & \mathrm{d}\tilde{\mathbf{k}}' \, \mathrm{d}\mathbf{q}_1\mathbf{q}_2 (2\pi)^3 \delta(\mathbf{k}' - \mathbf{q}_1 - \mathbf{q}_2) \, \mathrm{d}E' [\delta(\mathbf{q}_1 - \mathbf{q}_1')\delta(\mathbf{q}_2 - \mathbf{q}_2') \\ & + \delta(\mathbf{q}_1 - \mathbf{q}_2')\delta(\mathbf{q}_2 - \mathbf{q}_1')] f(E, E', \mathbf{k}) \, \mathrm{d}\tilde{\mathbf{q}}_1' \, \mathrm{d}\tilde{\mathbf{q}}_2' \\ & \times \delta(E' - \rho(\mathbf{q}_1') - \rho(\mathbf{q}_2'))\delta(\mathbf{k} - \mathbf{q}_1' - \mathbf{q}_2')|\mathbf{k}'\rangle. \end{aligned} \tag{58}$$

The integral in (58) is symmetric in the variables $\mathbf{q}_1$ and $\mathbf{q}_2$. Therefore, (58) is equal to

$$\begin{aligned} \frac{2\lambda}{\sqrt{2}} \int & \mathrm{d}\tilde{\mathbf{k}}' \, \mathrm{d}\mathbf{q}_1\mathbf{q}_2 (2\pi)^3 \delta(\mathbf{k}' - \mathbf{q}_1 - \mathbf{q}_2) \, \mathrm{d}E' \delta(\mathbf{q}_1 - \mathbf{q}_1')\delta(\mathbf{q}_2 - \mathbf{q}_2') \\ & \times f(E, E', \mathbf{k}) \, \mathrm{d}\tilde{\mathbf{q}}_1' \, \mathrm{d}\tilde{\mathbf{q}}_2' \delta(E' - \rho(\mathbf{q}_1') - \rho(\mathbf{q}_2'))\delta(\mathbf{k} - \mathbf{q}_1' - \mathbf{q}_2')|\mathbf{k}'\rangle \\ = \frac{2\lambda}{\sqrt{2}} \int & \mathrm{d}\tilde{\mathbf{k}}' \, \mathrm{d}\tilde{\mathbf{q}}_1 \, \mathrm{d}\tilde{\mathbf{q}}_2 \, \mathrm{d}E' \delta(\mathbf{k}' - \mathbf{q}_1 - \mathbf{q}_2)\delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) \\ & \times f(E, E', \mathbf{k})\delta(E' - \rho(\mathbf{q}_1') - \rho(\mathbf{q}_2'))|\mathbf{k}'\rangle. \end{aligned} \tag{59}$$

Now, using the identity

$$\delta(\mathbf{k}' - \mathbf{q}_1 - \mathbf{q}_2)\delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) = \delta(\mathbf{k} - \mathbf{k}')\delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) \tag{60}$$

and integrating over $\mathbf{k}'$, we finally get

$$\frac{\sqrt{2}\lambda}{2} \frac{1}{\omega(\mathbf{k})} \int \mathrm{d}E' f(E, E', \mathbf{k}) \left[\int \delta(E' - \rho(\mathbf{q}_1') - \rho(\mathbf{q}_2'))\delta(\mathbf{k} - \mathbf{q}_1 - \mathbf{q}_2) \right] |\mathbf{k}\rangle. \tag{61}$$

The expression between brackets in (61) is just $\tau(\epsilon^2 - \mathbf{k}^2)$ as in (44). Thus, (61) yields

$$\frac{\sqrt{2}\lambda}{2\omega(\mathbf{k})}\left[\int \mathrm{d}E'\, f(E, E', \mathbf{k})\tau(\epsilon^2 - \mathbf{k}^2)\right]|\mathbf{k}\rangle = V_{\mathrm{as}} \int \mathrm{d}E'\, f(E, E', \mathbf{k})|E', \mathbf{k}\rangle. \tag{62}$$

Thus, we can obtain $(H_{\mathrm{as}} - E)\Phi(E, \mathbf{k})$ using (50), (53)–(55) and (62). The result is of the form

$$(H_{\mathrm{as}} - E)\Phi(E, \mathbf{k}) = A(E, \mathbf{k})|\mathbf{k}\rangle + \int \mathrm{d}E'\, B(E, E', \mathbf{k})|E', \mathbf{k}\rangle \tag{63}$$

with

$$A(E, \mathbf{k}) = \left(\omega(\mathbf{k}) - E + \frac{\delta M^2}{\omega(\mathbf{k})}\right) f(E, \mathbf{k}) + \frac{\sqrt{2}\lambda}{2\omega(\mathbf{k})} \int \mathrm{d}E'\, \tau(E'^2 - \mathbf{k}^2) f(E, E', \mathbf{k}) \tag{64}$$

$$B(E, E', \mathbf{k}) = (E' - E) f(E, E', \mathbf{k}) + \lambda\sqrt{2}(2\pi)^3 f(E, \mathbf{k}). \tag{65}$$

Then, from (45) and (46), we obtain the following system of two equations for the functions $f(E, \mathbf{k})$ and $f(E, E', \mathbf{k})$:

$$A(E, \mathbf{k}) = 0 \qquad B(E, E', \mathbf{k}) = 0. \tag{66}$$

This system is analogous to that in the Friedrichs model [14, 15]. In order to solve the system given by (66), let us express $f(E, E', \mathbf{k})$ via $f(E, \mathbf{k})$ using (65) and the last identity in (66). The result is

$$f(E, E', \mathbf{k}) = A\delta(E' - E) - \frac{\lambda\sqrt{2}(2\pi)^3}{E' - E} f(E, \mathbf{k}) \tag{67}$$

where A is an arbitrary constant. Now let us use (67) in (64) and equate the result to zero. This gives the following equation for $f(E, \mathbf{k})$:

$$\left[\omega(\mathbf{k}) + \frac{\delta M^2}{\omega(\mathbf{k})} - E - \frac{\lambda^2(2\pi)^3}{\omega(\mathbf{k})} \int_{E_0}^{\infty} \mathrm{d}E' \frac{\tau(E'^2 - \mathbf{k}^2)}{E' - E}\right] f(E, \mathbf{k}) + A\frac{\lambda\sqrt{2}}{2\omega(\mathbf{k})}\tau(E^2 - \mathbf{k}^2) = 0. \tag{68}$$

The function τ is defined by (44) in the half-line $[E_0, \infty]$ with $E_0 = (4m^2 + \mathbf{k}^2)^{1/2}$ and this imposes the lower limit of integration in (68). The divergence of the integral due to the upper limit is cancelled by the counterterm δM^2:

$$Q(E, \mathbf{k}) = \lambda^2(2\pi)^3 \int_{E_0}^{\infty} \mathrm{d}E' \frac{\tau(E'^2 - \mathbf{k}^2)}{E' - E} - \delta M^2. \tag{69}$$

The counterterm δM^2 [16] is

$$\delta M^2 = \lambda^2(2\pi)^3 \int_{E_0}^{\infty} \mathrm{d}E' \frac{\tau(E'^2 - \mathbf{k}^2)}{E' - a}. \tag{70}$$

Therefore,

$$\begin{aligned} Q(E, \mathbf{k}) &= \lambda^2(2\pi)^3 \int_{E_0}^{\infty} \mathrm{d}E'\, \tau(E'^2 - \mathbf{k}^2)\left(\frac{1}{E' - E} - \frac{1}{E' - a}\right) \\ &= \lambda^2(2\pi)^3(E - a) \int_{E_0}^{\infty} \mathrm{d}E' \frac{\tau(E'^2 - \mathbf{k}^2)}{(E' - E)(E' - a)}. \end{aligned} \tag{71}$$

The usual 'on shell' renormalization corresponds to $a = \omega(\mathbf{k})$, which after (69) and (71), gives the zero value for the square brackets in (68). This choice is acceptable in the stable case, $M < 2m$, where the point $E = \omega(\mathbf{k})$ is smaller than the branch point E_0 of the function

$Q(E,\mathbf{k})$ and, therefore, lies outside the branch cut $[E_0,\infty)$ of $Q(E,\mathbf{k})$. With this choice, the subtraction constant a in (70) is real.

In the unstable case, characterized by $M > 2m$, the counterterm $\delta^2 M$ is analytic in a with a branch cut at $[E_0,\infty)$. Then, the choice $a=\omega(\mathbf{k})$ produces a counterterm with an infinite real part plus a nonvanishing imaginary part:

$$\int_{E_0}^{\infty}\frac{\tau(E'^2-\mathbf{k}^2)}{E'-\omega(\mathbf{k})\pm \mathrm{i}0} = PV\int_{E_0}^{\infty}\frac{\tau(E'^2-\mathbf{k}^2)}{E'-\omega(\mathbf{k})}\,\mathrm{d}E' \pm \mathrm{i}\pi\int_{E_0}^{\infty}\delta(E'-\omega(\mathbf{k}))\tau(E'^2-\mathbf{k}^2)\,\mathrm{d}E'$$
$$= \infty \pm \mathrm{i}\pi\tau(\omega^2(\mathbf{k})-\mathbf{k}^2) = \infty \pm \mathrm{i}\pi\tau(M^2) \qquad (72)$$

where PV stands for the Cauchy principal value.

In this case, the choice of the counterterm as in (70) leads to a complex Hamiltonian (as the counterterm is a part of the total Hamiltonian), which does not make sense. Therefore, we should make another choice for the counterterm in order to avoid infinities. Note that if $E > E_0$,

$$\int_{E_0}^{\infty}\frac{\tau(E'^2-\mathbf{k}^2)}{(E'-E)\pm \mathrm{i}0} = PV\int_{E_0}^{\infty}\frac{\tau(E'^2-\mathbf{k}^2)}{E'-\omega(\mathbf{k})} \pm \mathrm{i}\pi\tau(E^2-\mathbf{k}^2) \qquad (73)$$

which suggests a normalization for $Q(E,\mathbf{k})$ such that

$$\mathrm{Re}\,Q(E,\mathbf{k})|_{E=\omega(\mathbf{k})} = 0. \qquad (74)$$

This normalization can be produced by a real counterterm as it affects the real part of (69) only and is, therefore, acceptable. The function $Q(E,\mathbf{k})$ defined in this way is an analytic function with a cut on the real axis from the point $E_0 = (4m^2+\mathbf{k}^2)^{1/2}$ to infinity. Its discontinuity on the cut is

$$\frac{1}{2\mathrm{i}}[Q(E+\mathrm{i}0,\mathbf{k}) - Q(E-\mathrm{i}0,\mathbf{k})] = \pi\lambda^2(2\pi)^3\tau(E^2-k^2). \qquad (75)$$

Due to the normalization condition (74), the function in the square brackets in (68)

$$\eta(E,\mathbf{k}) = \omega(\mathbf{k}) - E - \frac{Q(E,\mathbf{k})}{\omega(\mathbf{k})} \qquad (76)$$

has no zeros on the real axis of the complex E-plane. Therefore the general solution of (66) is given by the following equation:

$$f(E,\mathbf{k}) = -A\frac{1}{\eta(E,\mathbf{k})}\frac{\lambda\sqrt{2}}{2\omega(\mathbf{k})}\tau(E^2-\mathbf{k}^2). \qquad (77)$$

Therefore, the second function $f(E,E',\mathbf{k})$ in (45) is also fixed:

$$f(E,E',\mathbf{k}) = A\left\{\delta(E'-E) + \frac{2\lambda^2(2\pi)^3}{2\omega(\mathbf{k})}\frac{\tau(E^2-\mathbf{k}^2)}{\eta(E,\mathbf{k})}\frac{1}{E'-E}\right\}. \qquad (78)$$

In both equations (77), (78) there is an ambiguity which arises due to the singular denominators $(E'-E)$. Consequently, we shall define two solutions of eigenvalue problem (46) that correspond to the incoming and the outgoing solution. The incoming and outgoing solutions are defined as boundary functions of analytic functions from above $(E+\mathrm{i}0)$ and from below $(E-\mathrm{i}0)$ the real axis respectively:

$$\Phi^{\mathrm{in}}_{\mathrm{out}}(E,\mathbf{k}) = A\left\{|E,\mathbf{k}\rangle + \frac{\lambda\sqrt{2}\cdot\tau(E^2-\mathbf{k})}{2\omega(\mathbf{k})\eta(E\pm \mathrm{i}0,\mathbf{k})}\right.$$
$$\left.\times\left[\lambda\sqrt{2}(2\pi)^3\int_{E_0}^{\infty}\mathrm{d}E'\frac{1}{E'-E\mp \mathrm{i}0}|E',\mathbf{k}\rangle - |\mathbf{k}\rangle\right]\right\}. \qquad (79)$$

This formula explicitly demonstrates that the energy spectrum of the asymptotic states in the eigenspace of N, corresponding to the eigenvalue 2, is absolutely continuous over the interval $[E_0, \infty)$ where $E_0 = (4m^2 + \mathbf{k}^2)^{1/2}$. The isolated eigenvalue associated with the one ψ-particle state $|\mathbf{k}\rangle$ has been dissolved to the continuum due to the interaction.

The partial resolvent of H_{as} is

$$\langle \mathbf{k}| \frac{1}{H_{\text{as}} - E} |\mathbf{k}\rangle = \frac{1}{\eta(E, \mathbf{k})} \tag{80}$$

where

$$\begin{aligned} \eta(E, \mathbf{k}) &= \omega(\mathbf{k}) - E - \frac{1}{\omega(\mathbf{k})} \cdot Q(E, \mathbf{k}) \\ &= \omega(\mathbf{k}) - E - \frac{1}{\omega(\mathbf{k})} \left[\operatorname{Re} Q(E, \mathbf{k}) + \mathrm{i} \frac{\lambda^2}{16\pi} \left(1 - \frac{4m^2}{E^2 - \mathbf{k}^2} \right)^{1/2} \theta(E - E_0(\mathbf{k})) \right]. \end{aligned} \tag{81}$$

We express the imaginary part of $Q(E, \mathbf{k})$ using (44) and (69). With the normalization condition (74), the real part of $\eta(E, \mathbf{k})$ vanishes for $E = \omega(\mathbf{k})$, which obviously is not a zero for $\eta(E, \mathbf{k})$, since the imaginary part is different from zero at $E = \omega(\mathbf{k})$.

4. The Gamow vectors

From (76), we see that the inverse partial resolvent $\eta(E, \mathbf{k})$ admits a continuation on the variable E, $\eta_+(E, \mathbf{k})$, from above to below through the branch cut $[E_0, \infty)$ and has a complex zero z_R on the lower half-plane. Now, we have to solve the equation $\eta(E, \mathbf{k}) = 0$, or equivalently:

$$E = \omega(\mathbf{k}) - \frac{1}{\omega(\mathbf{k})} \left[\operatorname{Re} Q(E, \mathbf{k}) + \mathrm{i} \frac{\lambda^2}{16\pi} \left(1 - \frac{4m^2}{E^2 - \mathbf{k}^2} \right)^{1/2} \theta(E - E_0(\mathbf{k})) \right]. \tag{82}$$

In the absence of interaction ($\lambda = 0$), $E = \omega(E, \mathbf{k})$. Therefore, we can write for small values of λ:

$$z_R = \omega(\mathbf{k}) + \lambda^2 z_1 + O(\lambda^4).$$

We replace E by z_R in equation (82) and expand both sides in terms of λ. Comparing the coefficients of λ^2, we have

$$z_1 = -\frac{\mathrm{i}}{8\pi} \left(1 - \frac{4m^2}{\omega^2(\mathbf{k}) - \mathbf{k}^2} \right)^{1/2} = -\frac{\mathrm{i}}{8\pi} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}.$$

Therefore,

$$z_R = \omega(\mathbf{k}) - \mathrm{i} \frac{M\Gamma}{2\omega(\mathbf{k})} \simeq (\mathbf{k}^2 + M^2 - \mathrm{i} M\Gamma)^{1/2} \tag{83}$$

where we have denoted

$$M\Gamma = \frac{\lambda^2}{8\pi} \left(1 - \frac{4m^2}{M^2} \right)^{1/2}. \tag{84}$$

This zero has a complex conjugate on the analytic continuation from above to below. Equation (83) demonstrates that the zero of $\eta(E, \mathbf{k})$ corresponds to a particle with complex square mass

$$M_c^2 = M^2 - \mathrm{i} M\Gamma \tag{85}$$

which is the remnant of a ψ-particle. The eigenstate (79) as a function of E has a pole at the point $E = E_c$ and its residue in this pole is the eigenstate with complex energy. Up to an irrelevant normalization constant this state is

$$\Phi^G(z_R, \mathbf{k}) = \lambda\sqrt{2}(2\pi)^3 \int_{E_0}^{\infty} \mathrm{d}E' \frac{1}{E' - z_R} |E', \mathbf{k}\rangle - |\mathbf{k}\rangle. \tag{86}$$

This Gamow vector [1, 4, 5] has a clear meaning as an antilinear functional on a suitably chosen space of test functions $\mathbf{\Phi}$. The test function space should be dense on the Hilbert space (39) of the states of two φ-particles and one ψ-particle, which is contained in the Fock space of the states of φ-particles and ψ-particles. Following the spirit of the construction for simple non-relativistic resonances [3–5, 17], a simple choice for the test function space, dense in the Hilbert space (39), is given by

$$[(\mathcal{H}^2_- \cap S) \otimes S(\mathbb{R}^3)] \oplus S(\mathbb{R}^3) \tag{87}$$

where

(1) $\mathcal{H}^2_-$ is the space of Hardy functions on the lower half-plane [18].
(2) S is the space of all real-valued complex functions that are differentiable at all orders such that they as well as their derivatives vanish at infinity faster than the inverse of any polynomial (the one-dimensional Schwartz space).
(3) $S(\mathbb{R}^3)$ represents the space of all complex functions on the three-dimensional real space R^3 having the same properties as S in (ii).

Then, the rigged Hilbert space is a triplet of the form $\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^\times$, where $\mathbf{\Psi}^\times$ is the dual space of $\mathbf{\Psi}$. Then, Gamow vectors belong to this dual $\mathbf{\Psi}^\times$.

A typical function in $\mathbf{\Phi}$ has the form $F(E, \mathbf{k}) + g(\mathbf{k}')$ where $F(E, \mathbf{k}) \in (\mathcal{H}^2_- \cap S) \otimes S(\mathbb{R}^3)$ and $g(\mathbf{k}') \in S(\mathbb{R}^3)$. The variables E and $\mathbf{k}$ in $F(E, \mathbf{k})$ represent respectively the total energy and the sum of the momenta for the two φ-particles. The variable $\mathbf{k}'$ is the momentum of the ψ-particle.

The action of the functional (86) on the function $K(E, \mathbf{k}'', \mathbf{k}') := F(E, \mathbf{k}'') + g(\mathbf{k}')$ is given by

$$\begin{aligned}\langle K(E, \mathbf{k}'', \mathbf{k}')|\Phi^G(z_R, \mathbf{k})\rangle &= \lambda\sqrt{2}(2\pi)^3 \int_{E_0}^{\infty} \mathrm{d}E' \frac{1}{E' - z_R} \langle F(E, \mathbf{k}'')|E', \mathbf{k}\rangle - \langle g(\mathbf{k}')|\mathbf{k}\rangle \\ &= \lambda\sqrt{2}(2\pi)^3 \int_{E_0}^{\infty} \mathrm{d}E' \frac{F(E, \mathbf{k})}{E' - z_R} + g(\mathbf{k}).\end{aligned} \tag{88}$$

For the fixed variable $\mathbf{k}$, the function $F(E, \mathbf{k})$ is a Hardy function and, therefore, the integral term of (88) converges. In the usual Fock space notation, a typical element of $\mathbf{\Phi}^\times$ can be written as

$$F(E, \mathbf{k}'')|E, \mathbf{k}''\rangle + g(\mathbf{k}')|\mathbf{k}'\rangle. \tag{89}$$

The functional $\Phi^G(z_R, \mathbf{k})$ has the following property:

$$H_{\mathrm{as}}\Phi^G(z_R, \mathbf{k}) = z_R \Phi^G(z_R, \mathbf{k}). \tag{90}$$

The proof of (90) goes as follows: we have seen that $\Phi(E, \mathbf{k})$ has a meromorphic extension from above to below on the variable E with a pole at the point z_R. Let us call this extension $\Phi_C(z, \mathbf{k})$. The residue of $\Phi_C(z, \mathbf{k})$ at z_R gives the Gamow vector $\Phi^G(z_R, \mathbf{k})$. On a neighbourhood of z_R one has the following expansion in terms of the complex energy z:

$$\Phi_C(z, \mathbf{k}) = \frac{1}{z - z_R} \Phi^G(z_R, \mathbf{k}) + \text{regular terms}. \tag{91}$$

The meromorphic extension of $\Phi(E, \mathbf{k})$ allows us to extend (46), so that we have

$$H_{\rm as}\Phi_C(z, \mathbf{k}) = z\Phi_C(z, \mathbf{k}) \tag{92}$$

for values of the complex energy z in the lower half-plane. If we bring (92) into (91), we get

$$\frac{H_{\rm as}\Phi^G(z_R, \mathbf{k})}{z - z_R} + \text{regular terms} = (z - z + z_R)\frac{\Phi^G(z_R, \mathbf{k})}{z - z_R} + \text{regular terms}. \tag{93}$$

If we identify the pole terms on the left- and right-hand sides of (93), we finally arrive at (90).

5. A second model

Let us start this section with a system of two particles without interaction. The corresponding state vector $\psi(x_1, x_2)$ has to fulfil the following pair of equations:

$$\left(p_1^2 - m^2\right)\varphi(x_1, x_2) = 0 \qquad \left(p_2^2 - m^2\right)\varphi(x_1, x_2) = 0 \tag{94}$$

where $p_i = \left(p_i^0, \mathbf{p}_i\right), i = 1, 2$ is the generator of the four-dimensional translation of coordinates $x_i = \left(x_i^0, \mathbf{x}_i\right)$. Instead of the coordinates x_i^μ and p_i^μ, $\mu = 0, 1, 2, 3$ and $i = 1, 2$, we may introduce the total and relative coordinates (resp. X^μ and q^μ) and the corresponding momenta (resp. P^μ and p^μ)

$$\begin{aligned} x_1^\mu &= X^\mu + \frac{1}{2}q^\mu \qquad & p_1^\mu &= \frac{P^\mu}{2} + p^\mu \\ x_2^\mu &= X^\mu - \frac{1}{2}q^\mu \qquad & p_2^\mu &= \frac{P^\mu}{2} - p^\mu. \end{aligned} \tag{95}$$

These operators have the following commutation relations:

$$[X^\mu, P^\nu] = [q^\mu, p^\nu] = -\mathrm{i}g^{\mu\nu} \tag{96}$$

all other commutators vanish. In terms of these new operators, equations (94) become

$$\left[P^2 - \left(4m^2 - p_\perp^2\right)\right]\psi(X_\mu, q_\mu) = 0 \tag{97}$$

$$[P \cdot p]\psi(X_\mu, q_\mu) = 0. \tag{98}$$

Equation (97) has the meaning of a mass shell condition with a squared mass operator $\mathfrak{m}^2$ defined by

$$\mathfrak{m}^2 := 4m^2 - p_\perp^2 \qquad p_\perp^\mu := p^\mu - P^\mu\frac{P \cdot p}{P^2}. \tag{99}$$

Due to (98), the system admits a one-time description [19, 20]. This system is the simplest one among the so-called Komar–Todorov systems [21]. In the most general case, the mass operator is a general function of (p^μ, q^μ), although the fact that the mass operator should commute with $P \cdot p$ implies that $\mathfrak{m}^2$ should have the following form:

$$\mathfrak{m}^2 = \mathfrak{m}^2\left(p_\perp^2, q_\perp^2, p_\perp q_\perp\right). \tag{100}$$

The one-time description for a two interacting relativistic particle system is not a trivial matter[5]. The validity of this description depends on the introduction of some extra conditions, such as for instance $P \cdot p = 0$.

The relativistic wave equation for the two-particle system is given as $\varphi(x_1, x_2)$ in terms of the old spacetime coordinates and by $\psi(X_\mu, q_\mu)$ in terms of the new coordinates. Let us define $q^2 := (x_1 - x_2)^2$. Then, we have the following Taylor expansion:

$$\psi(X_\mu, q_\mu) = \psi(X_\mu, q) + \frac{q_\alpha}{\sqrt{-q^2}}\psi^\alpha(X_\mu, q) + \frac{q_\alpha q_\beta}{\sqrt{-q^2}\sqrt{-q^2}}\psi^{\alpha\beta}(X_\mu, q) + \cdots \tag{101}$$

[5] In the classical context, see [19, 20].

where we have used the conventional notation in the summation of indices. For s-wave states, we may keep the term $\psi_0(X_\mu, q)$ only. The conjugate variable of q is given by

$$p = -\mathrm{i}\frac{\partial}{\partial q}.$$

In this approximation, the mass operator $\mathfrak{m}$ can be given by

$$\mathfrak{m}^2 := 4m^2 + p^2 = 4m^2 - \frac{\partial^2}{\partial q^2} \tag{102}$$

and the Klein–Gordon equation (97), in this approximation, is

$$\left(P^2 - 4m^2 + \frac{\partial^2}{\partial q^2}\right)\psi(X_\mu, q) = 0. \tag{103}$$

In (103), only the second derivative with respect to q appears. Therefore, even solutions of (103) in q exist. In order to simplify our problem, we are going to consider these kinds of solutions only, so that we are assuming that

$$\psi(X_\mu, -q) = \psi(X_\mu, q). \tag{104}$$

We are now in a position to solve equation (103) with condition (104). The solutions are of the following form:

$$\psi(X_\mu, q) = \int \mathrm{d}\kappa \int \frac{\mathrm{d}^3\mathbf{k}\cos\kappa q}{(2\pi)^4 2E(\mathbf{k},\kappa)}(B^*(\mathbf{k},\kappa)\exp(\mathrm{i}X\cdot k) + B(\mathbf{k},\kappa)\exp(-\mathrm{i}X\cdot k)) \tag{105}$$

where $k_\mu = (E, \mathbf{k})$ and

$$E(\mathbf{k},\kappa) = [4m^2 + \kappa^2 + \mathbf{k}^2]^{1/2}. \tag{106}$$

We change the variables in (105), in order to use E as a new independent variable instead of κ:

$$\kappa = (E^2 - \mathbf{k}^2 - 4m^2)^{1/2} \qquad \frac{\mathrm{d}\kappa}{E} = \frac{\mathrm{d}E}{\kappa}. \tag{107}$$

After (107), equation (105) reads

$$\psi(X_\mu, q) = \int_0^\infty \mathrm{d}E \int \frac{\mathrm{d}^3\mathbf{k}\cos\kappa(\tilde{\mathbf{k}}_\mu)q}{(3\pi)^4\kappa(k_\mu)}(B^*(\mathbf{k}, E)\exp(\mathrm{i}X\cdot k) + B(\mathbf{k}, E)\exp(-\mathrm{i}X\cdot k)). \tag{108}$$

Following the routine of second quantization, we replace the function $B(\mathbf{k}, E)$ by an operator, that we will also call $B(\mathbf{k}, E)$ for simplicity, and its complex conjugate $B^*(\mathbf{k}, E)$ by the adjoint operator $B^\dagger(\mathbf{k}, E)$, satisfying the following commutation relations:

$$[B(\mathbf{k}, E), B^\dagger(\mathbf{k}, E)] = (2\pi)^4\kappa(k_\mu)\delta^4(k_\mu - k'_\mu). \tag{109}$$

Thus, the function $\psi(X_\mu, q)$ becomes the operator

$$\psi(X_\mu, q) = \int_0^\infty \mathrm{d}E \int \frac{\mathrm{d}^3\mathbf{k}\cos\kappa(k_\mu)q}{(3\pi)^4\kappa(k_\mu)}(B^\dagger(\mathbf{k}, E)\exp(\mathrm{i}X\cdot k) + B(\mathbf{k}, E)\exp(-\mathrm{i}X\cdot k)). \tag{110}$$

Then, we have constructed the quantum bilocal field. It is important to remark that the solution $\psi(X_\mu, q)$ to our bilocal field somehow represents the state of two particles.

Now, we assume that this field interacts with another local quantum field $\varphi(X_\mu)$, which is the solution of a Klein–Gordon equation with point mass equal to M. The interaction is given by

$$H_{\mathrm{int}} = -\lambda \int \mathrm{d}^3\mathbf{x} \int_{-\infty}^{\infty} \mathrm{d}q\ \psi(X_\mu, q) f(q)\varphi(X_\mu). \tag{111}$$

The function $f(q)$ is called the form factor and can be chosen to be a smooth even function, as we assume in the following. If $\alpha(y)$ is the Fourier transform of $f(q)$ and $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$ are the respective annihilation and creation operators for the local field $\varphi(X)$, the interaction Hamiltonian is given by

$$P_0 = \int \frac{\mathrm{d}^3\mathbf{k}\,\mathrm{d}E}{(2\pi)^4\kappa(\mathbf{k},E)} E B^\dagger(\mathbf{k},E)B(\mathbf{k},E) + \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})}\omega(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k})$$
$$+ \int \frac{\mathrm{d}^3\mathbf{k}\,\mathrm{d}E}{(2\pi)^3 2\omega}\frac{\lambda\alpha(\kappa(\mathbf{k},E))}{\kappa(\mathbf{k},E)}(a(\mathbf{k}) + a^\dagger(-\mathbf{k}))(B^\dagger(\mathbf{k},E) + B(-\mathbf{k},E)) \qquad (112)$$

and the 3-momentum is

$$\mathbf{P} = \int \frac{\mathrm{d}^3\mathbf{k}\,\mathrm{d}E}{(2\pi)^4\kappa(\mathbf{k},E)}\mathbf{k}B^\dagger(\mathbf{k},E)B(\mathbf{k},E) + \int \frac{\mathrm{d}^3\mathbf{k}\,\mathbf{k}}{(2\pi)^3 2\omega(\mathbf{k})}a^\dagger(\mathbf{k})a(\mathbf{k}) \qquad (113)$$

with

$$\omega(\mathbf{k}) = (\mathbf{k}^2 + M^2)^{1/2}. \qquad (114)$$

Remark. The bilocal field results as an approximation of the behaviour of two relativistic interacting particles. The interaction of the bilocal field with the field $\varphi(x)$ represents the interaction of the two former particles with a third of fixed mass M. This is therefore a model that approaches the behaviour of three interacting particles, just like the other one described in the preceding sections.

The next step is to *diagonalize* the 4-momentum (112) and (113). This means that we are looking for creation, $b^\dagger(E,\mathbf{k})$, and annihilation, $b(E,\mathbf{k})$, operators such that the 4-momentum components P_μ can be written as

$$P_\mu = \int \frac{\mathrm{d}^3\mathbf{k}\,\mathrm{d}E}{(2\pi)^4\kappa(E,\mathbf{k})}k_\mu b^\dagger(E,\mathbf{k})b(E,\mathbf{k}) \qquad (115)$$

where κ is given in (107). To achieve it, we pose the eigenvalue equation

$$[P_\mu, b^\dagger(E,\mathbf{k})] = k_\mu b^\dagger(E,\mathbf{k}) \qquad (116)$$

where $k_\mu = (E,\mathbf{k})$ and the P_μ are given by (112), (113). To solve (116), i.e., to obtain the creation operators $b^\dagger(E,\mathbf{k})$, we make the following ansatz:

$$b^\dagger(E,\mathbf{k}) = \int \mathrm{d}E'(T(E,E',\mathbf{k})B^\dagger(E',\mathbf{k}) + R(E,E',\mathbf{k})B(E',-\mathbf{k}))$$
$$+ t(E,\mathbf{k})a^\dagger(\mathbf{k}) + r(E,\mathbf{k})a(-\mathbf{k}) \qquad (117)$$

which obviously means that we are assuming that $b^\dagger(E,\mathbf{k})$ is a linear combination of $B^\dagger(E,\mathbf{k})$, $B(E,\mathbf{k})$; $a^\dagger(\mathbf{k})$ and $a(\mathbf{k})$. This problem was solved in [1]. The coefficients $T(E,E',\mathbf{k})$, $R(E,E',\mathbf{k})$, $t(E,\mathbf{k})$ and $r(E,\mathbf{k})$ depend on the form factor $f(q)$ in (111) and can be written in terms of the Green function [1]

$$G(E,\mathbf{k}) = \frac{1}{\omega^2 - E^2 - \Pi(E,\mathbf{k})} \qquad (118)$$

with

$$\Pi(E,\mathbf{k}) = \int_{E_0}^{\infty} \mathrm{d}E'\,2E'\frac{\rho(E',\mathbf{k})}{E'^2 - E^2} \qquad (119)$$

and $E_0 = (4m^2 + \mathbf{k}^2)^{1/2}$. Therefore, $G(E,\mathbf{k})$ depends on

$$\rho(E,\mathbf{k}) = 2\pi\frac{\lambda^2\alpha^2(\kappa(E,\mathbf{k}))}{\kappa(E,\mathbf{k})} \qquad (120)$$

through $\Pi(E, \mathbf{k})$, since $\rho(E, \mathbf{k})$ is given by equation (120). Note that $\rho(E, \mathbf{k})$ depends on $\alpha(y)$ which is the Fourier transform of the form factor $f(q)$.

From (120), we see that we cannot choose the form factor $f(q)$ arbitrarily. For instance, if we fix $f(q) \equiv 1$, its Fourier transform $\alpha(y)$ is a Dirac delta that cannot be squared thus making (120) meaningless. To be on the safe side, we may choose the form factor $f(q)$ to be a smooth (i.e., Schwartz) function.

For $\mathbf{k}$ fixed, the function $G(E, \mathbf{k})$ is a function of the complex variable E with a cut on the real semiaxis $[E_0, \infty)$. The boundary values of the complex variable function $G(E, \mathbf{k})$ from above to below and from below to above are respectively given by $G_+(E, \mathbf{k})$ and $G_-(E, \mathbf{k})$.

The coefficients $T(E, E', \mathbf{k}), R(E, E', \mathbf{k}), t(E, \mathbf{k})$ and $r(E, \mathbf{k})$ in (117) depend on $G(E, \mathbf{k})$ and hence on the operator $b^\dagger(E, \mathbf{k})$ and its adjoint $b(E, \mathbf{k})$. If we use $G_+(E, \mathbf{k})$ instead of $G(E, \mathbf{k})$, we are using the so-called *incoming boundary conditions*. Henceforth, we shall denote by $b^\dagger_{\mathrm{in}}(E, \mathbf{k})$ and $b_{\mathrm{in}}(E, \mathbf{k})$ the solutions of (117) and its adjoint equation, where we have used $G_+(E, \mathbf{k})$ instead of $G(E, \mathbf{k})$. (If instead of $G_+(E, \mathbf{k})$, we use $G_-(E, \mathbf{k})$, we obtain the new solution $b^\dagger_{\mathrm{out}}(E, \mathbf{k})$ and $b_{\mathrm{out}}(E, \mathbf{k})$ corresponding to the *outgoing boundary conditions*.)

Now, there are two possible situations. If $M < 2m$, the functions $G_+(E, \mathbf{k})$ and $G_-(E, \mathbf{k})$ are analytic with no singularities on the upper and lower half-planes respectively [1]. On the other hand, if $M \geqslant 2m$, $G_+(E, \mathbf{k})$ has a pole at $E^2 = \mathbf{k}^2 + \mu_c^2$ and $G_-(E, \mathbf{k})$ has a pole at $E^2 = \mathbf{k}^2 + \mu_c^{*2}$, where the star denotes complex conjugation, and

$$\mu_c^2 = \mu^2 - \mathrm{i}\mu\Gamma. \tag{121}$$

The real and positive numbers μ and Γ depend on the form factor $f(q)$.

The existence of a pole of this kind implies the presence of a metastable state (resonance) of the system [1]. Carrying the analogy with the previous model further, the metastable state appears when the particle of mass M can decay into the two-particle system described by the bilocal field. The condition $M \geqslant 2m$ is of course necessary for this process to take place.

The equation for the complex pole μ_c is [1]

$$\omega^2(\mathbf{k}) - E^2 - \int \mathrm{d}E'^2 \frac{\rho(E', \mathbf{k})}{E'^2 - E^2} = 0. \tag{122}$$

For small values of the coupling constant λ, we find [1]

$$\mu_c^2 = M^2 - 2\mathrm{i}\pi^2\lambda^2 \frac{\left[\alpha(\sqrt{M^2 - 4m^2})\right]^2}{\sqrt{M^2 - 4m^2}}. \tag{123}$$

In order to obtain the Gamow vectors for this resonance, we have first to obtain the vacuum Ω being annihilated by the operators $b(E, \mathbf{k})$. This can be obtained by the vacuum $|0\rangle$, annihilated by $B(E, \mathbf{k})$ and $a(\mathbf{k})$, i.e., the initial vacuum state defined by

$$B(E, \mathbf{k})|0\rangle = 0 \qquad a(\mathbf{k})|0\rangle = 0. \tag{124}$$

The new vacuum Ω is obtained from the old vacuum $|0\rangle$ by a Bogolubov transformation [1]:

$$\Omega = \mathrm{e}^V |0\rangle \tag{125}$$

where V depends on the creation operators $B^\dagger(E, \mathbf{k})$ and $a^\dagger(\mathbf{k})$. The new vacuum Ω has the following property:

$$b(E, \mathbf{k})\Omega = 0. \tag{126}$$

Now, let us define

$$\Phi_{\mathrm{in}}(E, \mathbf{k}) = b^\dagger_{\mathrm{in}}(E, \mathbf{k})|\Omega\rangle \tag{127}$$

which, as a function of the complex variable E, has a pole in the analytic continuation from above to below. This pole is located at the point $z_R = (\mathbf{k}^2 + \mu^2 - \mathrm{i}\mu\Gamma)^{1/2}$ [1]. In a neighbourhood of z_R, the vector $\Phi_{\mathrm{in}}(E, \mathbf{k})$ has the form

$$\Phi_{\mathrm{in}}(E, \mathbf{k}) = \frac{1}{E - z_R}\varphi_{\mathrm{in}}^G(\mathbf{k}) + \text{regular part}. \tag{128}$$

By construction, $\varphi_{\mathrm{in}}^G(\mathbf{k})$ is the Gamow vector associated with the resonance pole z_R. Mathematically, $\varphi_{\mathrm{in}}^G(\mathbf{k})$ can be rigorously defined as a functional on a space of test vectors, dense in the Fock space [1], so that the following properties hold:

(1) The Gamow vector $\varphi_{\mathrm{in}}^G(\mathbf{k})$ is a eigenvector of the Hamiltonian with eigenvalue z_R, i.e.,

$$P_0\varphi_{\mathrm{in}}^G(\mathbf{k}) = z_R\varphi_{\mathrm{in}}^G(\mathbf{k}). \tag{129}$$

(2) It decays exponentially (in a weak sense, note that $\varphi_{\mathrm{in}}^G(\mathbf{k})$ is not in the Fock space), so that if $t \geqslant 0$,

$$\exp(-\mathrm{i}tP_0)\varphi_{\mathrm{in}}^G(\mathbf{k}) = \exp(-\mathrm{i}tz_R)\varphi_{\mathrm{in}}^G(\mathbf{k}). \tag{130}$$

Thus, the behaviour of the Gamow vector $\varphi_{\mathrm{in}}^G(\mathbf{k})$ is analogous to the behaviour of decaying Gamow vectors in nonrelativistic systems [1, 3–5, 10].

6. Concluding remarks

In this paper, we have presented two models of interacting fields that could be exactly solved in the three-particle approximation. We are assuming that one particle decays into the other two, which are supposed to be identical for simplicity. In both cases further approximations are required. In the first case, it is ultraviolet renormalization and in the second case, we choose an approximation when dealing with the interaction between two particles. Once we have made these approximations, the solution obtained is exact, because after the approximations both models can be treated as generalized Friedrichs models, which are exactly solvable [1, 6, 14].

These two models have a very important feature in common. When the relation between the masses of the three particles involved allows, the system becomes unstable. We describe these unstabilities and construct the respective Gamow vectors for unstable states.

We completely develop the first model in detail. The second model has been treated elsewhere as an interaction between a local and a bilocal field without any reference to the origin and motivation of such a construction [1]. In the present paper, we give and explain this motivation, which is based on an approximation of the relativistic interaction between three particles.

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